THE ELECTROHYDROSTATICS OF A CONDUCTIVE LIQUID MENISCUS[1]

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Abstract - Electric fields offer a convenient way to manipulate liquid surfaces. The electrical forces produced are perpendicular to the surface of a conducting liquid, but unlike pressure, act only on the surface itself. The opposing force is surface tension. Self-consistent solutions for the shapes of axisymmetric menisci in electric fields are derived for conductive liquids with constant surface tension. In a uniform field, the shapes which result are rounded cones. The dimensionless quantity $\varepsilon_o E^2 b/\sigma$, where b is the radius of the hole surrounding the meniscus and σ is the surface tension, is a key parameter in the models.

INTRODUCTION

Many modern applications of electrostatics involve the manipulation of liquid droplets or surfaces. In applications such as liquid toner electrophotography and ink jet printing, the ability to control precisely a liquid surface would be valuable. The issue of how a static liquid surface interacts with an electric field is germane to these applications. The static case can provide insight into the effects of surface tension and geometrical parameters. It can reveal the conditions under which electrostatic forces dominate. In particular, it is well known that at some critical field strength, a liquid surface becomes unstable and emits droplets or jets of charged fluid. Solutions for the static case should be useful in predicting this stability limit.

An early attempt to take advantage of such electrohydrostatic field effects was explained in the seminal electrostatic ink jet patent by Winston [1]. There, an electric field was to draw liquid from an orifice and direct it toward a paper target. Soon afterward, Taylor [2] presented an analysis of the electrohydrostatic stability limit of a particular liquid surface shape. He demonstrated that a cone was a possible meniscus shape under specific electric field geometries, but that only one particular conical shape met all boundary conditions. This "Taylor cone" of apex angle 98.3° has been approached in various experiments [2], [3]. Davey and Melcher [4] examined the behavior of flat surfaces under the influence of electric and magnetic fields. Joffre and his coworkers [3], [5], studied the effect of electric fields on meniscus shape.

In the work of Joffre, both theoretical and experimental results for meniscus shapes in electric fields are presented. The electrostatic and hydrostatic equations are solved simultaneously to predict meniscus shape. The maximum meniscus extension is presented as a function of voltage, surface tension, hydrostatic pressure, orifice diameter, and electrode spacing for a tube-plane geometry. While the shape of a meniscus is treated in [3], the issue of surface instability and its relation to fluid properties is covered only briefly, and the breakdown limit of the air surrounding the meniscus is not taken into account. The basic question of whether an electric field is strong enough for active control of a liquid surface is not addressed.

The objective of the present study is to extend the work of Joffre in order to gain insight into practical aspects of meniscus electrohydrostatics. Of special interest is the way in which fluid parameters influence the fields.

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ELECTROHYDROSTATIC FIELD THEORY

When an electrically conductive liquid is exposed to an electric field, the field will be normal to the liquid surface and will exert an attractive force on that surface. Maxwell's stress tensor [6] can be used to express the force per unit area as a pressure $P_e = \varepsilon E^2/2$, where ε is the permittivity of the fluid in the electric field gap, and E is the electric field at the liquid surface. Under static conditions, this electric pressure must balance any hydrostatic pressure in the liquid, the liquid surface tension forces, and any other forces. The problem is to find the meniscus shape corresponding to such a force balance and identify the associated fields.

Approach to the Problem

Surface tension forces can be expressed in terms of curvature. A differential equation is derived for the curvature of a meniscus, under the influence of some given electric pressure and hydrostatic pressure. The electrostatic equation has the equipotential corresponding to the position of the meniscus as one boundary condition. These two equations must be solved simultaneously to yield a self-consistent result for the meniscus shape.

Assumptions are as follows:

-- The only forces acting on the meniscus are surface tension, hydrostatic pressure, and electric pressure.

-- The meniscus is completely stationary, and there is no internal liquid motion.

-- Surface tension is assumed to be a known constant, as is any applied hydrostatic pressure.

-- A single value of hydrostatic pressure is assumed at all meniscus points.

-- Singularities in the electric field are not permitted. This is expressed in the method by requiring that the meniscus have a non-singular curvature in all directions.

Surface Force Balance

Figure 1 shows the geometry of an element of a meniscus surface. The y axis corresponds to the vertical direction. In the Figure, f_{net} is the net force from electric fields and hydrostatic pressure. Surface tension creates forces in each of the four directions along the surface. In the static case, each component of the forces must balance. The vertical balance leads to

$$f_{nety} + f_{\sigma 1y} + f_{\sigma 2y} + f_{\sigma 3y} + f_{\sigma 4y} = 0 \quad . \qquad (1)$$

The form of terms f_σ in (1) is well known: surface

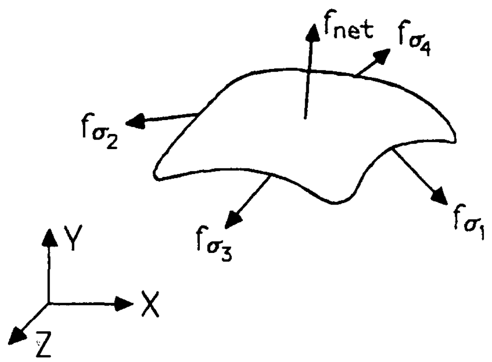

Figure 1. A Differential Element of a Meniscus

tension multiplied by the three-dimensional curvature $(1/R_1 + 1/R_2)$ and the element area gives this opposing force. Thus a pressure balance corresponding to the force can also be written as

$$P_{net} = -\sigma(\frac{1}{R_1} + \frac{1}{R_2}) \tag{2}$$

where R_1 is the radius of curvature along one tangent direction and R_2 is the radius of curvature along the direction normal to that of R_1. Equation (2) suggests a normalized pressure term $P_{net}R/\sigma$, where σ/R is a "surface tension pressure." This dimensionless ratio would represent the relative strength of electric forces in comparison with surface tension if P_{net} were equal to P_e.

Any externally applied hydrostatic pressure will manifest itself directly at the meniscus. Therefore, the net pressure P_{net} is given as

$$P_{net} = P_e(x,y,z) + P_{stat} \tag{3}$$

where $P_e(x,y,z)$ is the electric pressure at any point, and P_{stat} is some imposed hydrostatic pressure. Let P_e be normalized as $P_{emax}\ F(x,y,z)$, where P_{emax} is the highest electric field in the system (generally at the tip of the meniscus). The function $F(x,y,z)$ is a "distribution function" such that $0 < F < 1$: F represents the geometric dependence of P_e. Equations (2) and (3) represent very general relationships which must be satisfied in any static case.

If the meniscus is immersed in a homogeneous dielectric of permittivity ϵ, the electrostatic potential ϕ is given by

$$\nabla^2\phi = 0 \tag{4}$$

so that at any point on the meniscus,

$$P_e(x,y,z) = \epsilon(\nabla\phi)^2/2 \tag{5}$$

Equations (2) and (5) must be solved simultaneously to yield meniscus shape. The solution is performed iteratively, as described below.

Let us examine two simple geometries to check the basic validity of the force balance equations. In the case of a meniscus confined to a circular orifice of radius b, it is well known that a particular value of uniform static pressure P_{stat} will produce a meniscus which is a section of a sphere with radius $r \geq b$. The pressure balance becomes

$$P_{stat} = -\sigma(\frac{1}{r} + \frac{1}{r}) \tag{6}$$

which gives $Pr/\sigma = 2$. This is indeed the correct expression for tension in the skin of a thin-walled spherical vessel, as can be found in any standard mechanics text [7]. A second case is that of a cone, as in Figure 2. Curvature $1/R_1$ is zero, while radius

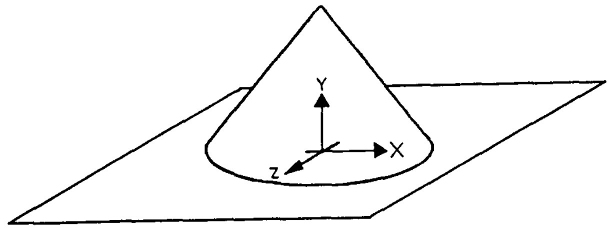

Figure 2. Geometry of a Conical Meniscus

R_2 is a constant times x. The pressure balance becomes

$$P_e + P_{stat} = -\sigma(\frac{1}{kx}) \tag{7}$$

In the case of $P_{stat} = 0$, this implies that a cone is possible when $P_e \propto 1/x$, which coincides with Taylor's results [2]. It is interesting to notice that at the tip of the cone, both curvature and electric field will be infinite. In principle, the two can balance out, although such a situation would not be realistic.

A General Sample Case

Consider a fairly general problem: a meniscus, in air, confined to a circular hole of radius b in one plate of a large parallel plate capacitor, as shown in Figure 3.

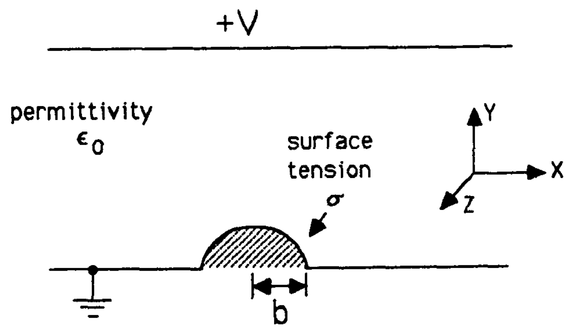

Figure 3. Axisymmetric Meniscus in Uniform Field

The problem has axial symmetry, and also has a relatively simple field geometry. The meniscus can be defined by a curve in the x-y plane, with equation $y = f(x)$. In spherical coordinates, the curve has an equation $r = g(\theta)$. Surface curvature can be written in spherical coordinates as [8]

$$(\frac{1}{R_1} + \frac{1}{R_2}) = \frac{[\frac{1}{r} + \frac{2}{r^3}(\frac{dr}{d\theta})^2 - \frac{1}{r^2}(\frac{d^2r}{d\theta^2})] + \frac{1}{r}[1 - \frac{dr/d\theta}{r\tan\theta}][1 + (\frac{dr}{d\theta})^2]}{[1 + \frac{1}{r^2}(\frac{dr}{d\theta})^2]^{3/2}} \tag{8}$$

minimum orifice radius which will permit electrostatic fields to cause a flow. For example, in the pure electrostatic case with $P_{stat} = 0$, the minimum orifice radius is $4.05\sigma/39.8$, which is tabulated for three liquids in Table I. The minimum radii are more like

TABLE I

MINIMUM ORIFICE RADII FOR ELECTROSTATIC LIQUID EJECTION IN AIR, $P_{stat} = 0$

Liquid	Surface tension, N/m	minimum radius
water	0.072	7.32 mm
methanol	0.022	2.24 mm
ethylene glycol	0.048	4.88 mm

pipe sizes than like ink jet orifice radii. For water, the diameter is more than 1/2"!

Clearly, there is a practical problem with exclusively electrostatic meniscus control. Consider instead the case where a hydrostatic pressure is also applied to the liquid. Then the electric field is an "ejection aid" rather than the sole controlling force. For example, if $P_{stat}b/\sigma$ is set to 1.97 (near the hydrostatic stability limit), a ratio $P_{emax}b/\sigma$ of only 0.058 will create instability. Table II shows the maximum $P_{emax}b/\sigma$ for which convergence could be achieved, at four different values of $P_{stat}b/\sigma$. Also given are the corresponding initial (zero-field) meniscus heights and the heights at the convergence limit, normalized to radius b. Meniscus shapes for the three nonzero P_{stat} cases are shown in Figure 5.

TABLE II

MINIMUM $P_{emax}b/\sigma$ FOR ELECTROSTATIC LIQUID EJECTION IN AIR, $P_{stat} \neq 0$

$P_{stat}b/\sigma$	$P_{emax}b/\sigma$ at stability limit	Initial height	Height at stability limit
0.0	4.05	0.0	0.54
1.15	1.72	0.315	0.75
1.73	0.51	0.577	0.88
1.97	0.058	0.839	0.97

In Figure 6, the limiting $P_{emax}b/\sigma$ values are plotted versus $P_{stat}b/\sigma$. The dashed line is a linear interpolation between the data points. The shaded region, for which the menicus shows static stability, can be summarized simply as

$$\frac{P_{stat}b}{\sigma} + \frac{P_{emax}b}{2\sigma} < 2 \ .$$

Minimum orifice sizes can be derived from the limiting value of $P_{emax}b/\sigma$ as before. Table III lists minimum orifice radii for instability at various $P_{stat}b/\sigma$ values, for the liquids of Table I. The maximum tolerable value of surface tension pressure is also tabulated.

The Physical Experiment

The setup consists of a small stainless steel nozzle, with outside diameter 230 μm, which protudes 470 μm from a backplate. The nozzle is held 600 μm from a flat field plate. The backplate is

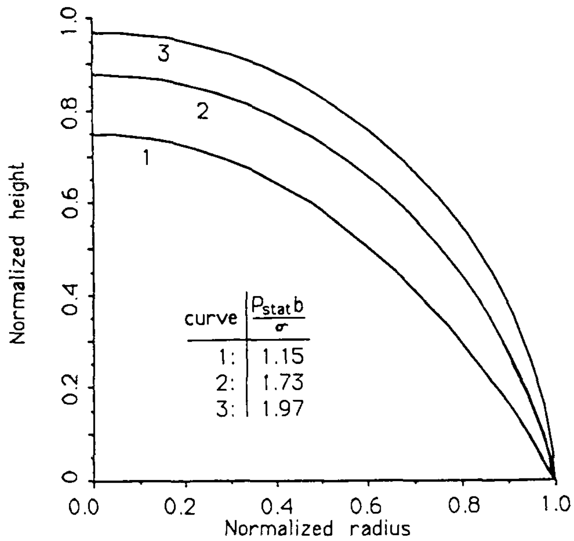

Figure 5. Meniscus Shapes at Stability Limit

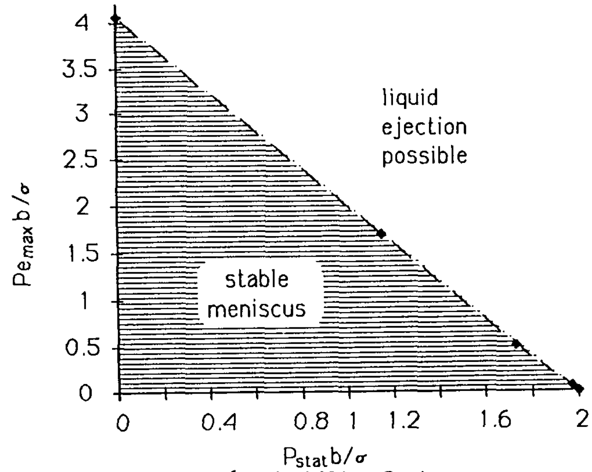

Figure 6. Stability Regimes

TABLE III

MINIMUM ORIFICE RADII FOR ELECTROSTATIC LIQUID EJECTION IN AIR, $P_{stat} \neq 0$

$P_{stat}b/\sigma$	$P_{emax}b/\sigma$ at stability	σ/b (Pa) at air breakdown	minimum radius (mm) water	ethylene glycol	methanol
0.0	4.05	9.8	7.2	4.8	2.2
0.5	3.0	13.3	5.4	3.6	1.7
1.0	2.0	19.9	3.6	2.4	1.1
1.15	1.72	23.4	3.1	2.07	0.94
1.5	1.0	39.8	1.8	1.21	0.55
1.73	0.51	78.1	0.923	0.62	0.28
1.97	0.058	685.	0.105	0.070	0.032

approximately 1 mm from the field plate. The field plate has a 400 μm hole opposite the nozzle to catch any liquid jetting from the meniscus. The nozzle is held at ground potential, while the field plate is supplied from a high voltage source through a current

By rearranging terms and substituting the balance relation from (2) above, (8) becomes

$$\frac{d^2\rho}{d\theta^2} = [\frac{2}{\rho^2}(\frac{d\rho}{d\theta})^2+1]\rho + \rho[1-\frac{d\rho/d\theta}{\rho\tan\theta}][1+\frac{1}{\rho^2}(\frac{d\rho}{d\theta})^2]$$

$$-\rho^2[\frac{P_{emax}b}{\sigma}F(\theta) + \frac{P_{stat}b}{\sigma}][1+\frac{1}{\rho^2}(\frac{d\rho}{d\theta})^2]^{3/2} \quad (9)$$

where ρ is the normalized radius r/b. All physical parameters are reflected in the dimensionless ratios $P_{emax}b/\sigma$ and $P_{stat}b/\sigma$.

Unfortunately, (9) is of second order, so that the boundary condition which requires the meniscus to contact the electrode surface does not fully specify the problem. A second condition is the requirement of nonsingular curvature, which implies that $d\rho/d\theta = 0$ at the axis (i.e. the meniscus has a horizontal tangent plane at its tip). The two conditions are awkward, since they are specified at a different locations. To cope with the difficulty, a so-called shooting method was used: a value of $d\rho/d\theta$ at the meniscus edge is guessed. Equation (9) is then solved iteratively until $d\rho/d\theta = 0$ at the axis x = 0.

FIELD SOLUTIONS FOR SAMPLE CASE

Basic Iterative Procedure

The electric field function F(θ) in (9) is, of course, also a function of shape, so that (9) is an implicit equation. For iterative solution, the problem is discretized and the following procedure is used:

1. For a given P_{stat}, find the meniscus shape for $P_e = 0$. This meniscus intersects the plate with a definite contact angle α_c, which gives a tentative value for $d\rho/d\theta$ at the meniscus edge.

2. Given the meniscus shape, solve the Laplace equation and find the normalized electric pressure F(θ) at surface points. A standard finite element package, ANSYS (Swanson Analysis Systems Inc.) was used for the results presented here.

3. Perform a fourth-order polynomial fit to the finite element results in order to obtain a continuous function which represents the electric pressure along the surface.

4. Integrate (9) with the fourth-order Runge-Kutta method to find a tentative meniscus outline curve.

5. As described above, the nonsingular field assumption requires that $d\rho/d\theta = 0$ at the y axis. If this condition is not met, adjust the ratio $P_{emax}b/\sigma$ and repeat the integration of step 3. This is the iterative process for solving (9), as described above. Contact angle α_c will be different from the initial value.

6. If the shape has changed very little, assume that a solution has been found. Otherwise, return to step 2.

The final result is a meniscus shape corresponding to a chosen value of $P_{stat}b/\sigma$ and a specific value of $P_{emax}b/\sigma$.

Results

When $P_{emax}b/\sigma$ is zero, a stable meniscus will form whenever

$$0 < P_{stat}b/\sigma < 2 \quad (10)$$

To test the numerical methods, a value of $P_{stat}b/\sigma$ was set, and an erroneous contact angle α_c was used to begin the iteration scheme. In all cases, meniscus shape converged to the correct spherical section.

A more interesting case appears when $P_{stat}b/\sigma$ is zero. Then the shape is determined only by the electric field value $P_{emax}b/\sigma$. Shapes for several values of $P_{emax}b/\sigma$ are shown in Figure 4. Naturally, the

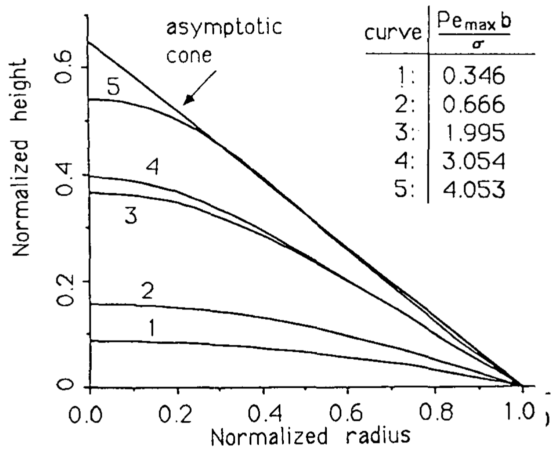

Figure 4. Meniscus Shape Outlines with $P_{stat} = 0$.

higher the field, the more the meniscus extends from the orifice. Above $P_{emax}b/\sigma = 4.05$, convergence could not be achieved. Presumably, this corresponds to the surface stability limit: if the ratio exceeds 4.05, liquid should be ejected from the surface.

Instability in the numerical solution is related to the appearance of reverse curvature of the meniscus. If more iterations are performed, the meniscus grows without bound. This is a psuedo-physical process in which meniscus extension increases the electric field enhancement, and the meniscus extends and sharpens its curvature in attempting to restore the static balance.

It is interesting that a ratio of 4.05 is far higher than the limiting ratio of $P_{stat}b/\sigma$: a meniscus can support twice as much peak electric pressure as hydrostatic pressure because the electric pressure is not uniform. Notice that while the solution appears to approach a conical shape, it is still quite rounded. The asymptotic cone shown in Figure 4 has an apex angle of 122° -- quite different from the Taylor cone. This might be caused by the truncated cone in a uniform field, which differs from Taylor's system of an infinite cone in a properly chosen field. It might also reflect Taylor's static results: he not-iced the 98.6° cone only when the meniscus was actually in motion, and saw shallower angles on stationary menisci.

For a given P_{stat}, the maximum stable meniscus height case has a corresponding value of $P_{emax}/b\sigma$ which can be regarded as the minimum necessary to pull out droplets from the meniscus. Assuming that E cannot exceed the atmospheric breakdown strength of about 3 MV/m, P_{emax} cannot exceed $\varepsilon_o\ (9.10^{12})/2$, or 39.8 Pa. Given the minimum $P_{emax}b/\sigma$, with P_{emax} limited to the breakdown level $P_{ebk} = 39.8$ Pa, this implies a minimum b/σ ratio to reach instability, or conversely a maximum σ/b (surface tension pressure) that can be tolerated without flow. For a specific liquid, this leads to a limiting resistor. An anti-wetting agent. Rain-X (a registered trademark of Unelko Corp., Scottsdale, Arizona), is applied to the outside of the nozzle in operation. The meniscus is observed and photographed through a long-working-distance stereo microscope. A photograph of a meniscus and the setup, taken at 50X magnification, appears as Figure 7.

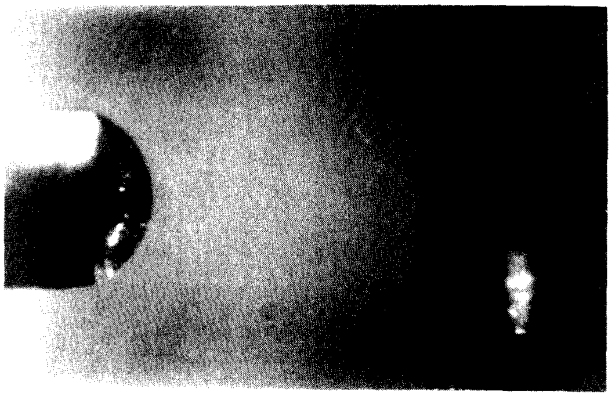

Figure 7. Photograph of Experimental Setup at 50X

A preliminary experiment was performed with methanol. The methanol was filtered to avoid clogging the nozzle. The accepted value of surface tension is 0.022 N/m, and density is 790 kg/m^3. The liquid is supplied to the nozzle from a small reservoir. P_{stat} is controlled by adjusting the reservoir height. For methanol with this nozzle, the surface tension pressure σ/b is 190 Pa. The expected hydrostatic limit is twice the surface tension pressure, or 380 Pa. The pressure head of a methanol column is 7.75 Pa/mm, which implies a hydrostatic limit for this nozzle of 49 mm head. The measured value of 50 ± 2 mm is in good agreement with this.

In this case, the breakdown pressure ratio limit $P_{ebk}b/\sigma = 0.21$. From the graph of Figure 6, this should require $P_{stat}b/\sigma$ of 1.8, or P_{stat} of 340 Pa (44.1 mm head) in order to destabilize the meniscus electrically. Actual results showed that a sudden applied voltage of 2.0 kV was able to destabilize the meniscus with a head as low as 29 mm, significantly less than the expected value. Vibration, wetting of the outside of the nozzle, or nonuniformities in the nozzle tip could contribute to the discrepancy. For example, the Rain-X coating appears to be compromised after it is wetted by methanol. The results also showed that a near-conical meniscus could be achieved at very high fields, although the tip was so fine that it is not clear whether the shape was static or involved a very fine jet flow.

CONCLUSION

Implications of Theoretical Results

In the context of an electrostatic ink jet or similar application, the above results give some significant insights into design parameters:

-- Surface forces obtained with electrostatic fields are small, with electric pressures below about 40 N/m^2.

-- Electric field forces are balanced by surface tension. The lower the surface tension, the easier it is to apply electrostatic forces.

-- Water is not a good choice for working fluid in an electrostatic liquid control application because of its high surface tension. Alcohols have much lower surface tension, and should be a better choice.

-- Electric forces benefit from a hydrostatic bias force, set to allow a small electrostatic force to overcome surface tension. The required hydrostatic heads are small. For example, methanol operating from an orifice with b = 0.28 mm and $P_{stat}b/\sigma = 1.73$ requires static pressure of 136 N/m^2, which means a head of 1.75 cm of methanol.

Future Work

Extensive experiments will be performed to test the theoretical results. The static case is a prelude to more comprehensive dynamic results. The effects of jet flows, meniscus vibrations, and dynamic electric field pulses all need to be studied for insight into practical applications of electrostatic meniscus control.

An electric field can control a liquid meniscus under certain conditions. When control is performed in air, only limited forces can be obtained. These low forces require an added hydrostatic pressure in order to cause surface instability.

Low surface tension liquids, such as alcohols, are better for control applications than water.

ACKNOWLEDGEMENT

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