Computer-Aided Analytical Solutions of Laplace's Equation

Joseph M. Crowley
Electrostatic Applications
16525 Jackson Oaks Drive
Morgan Hill, California 95037
(408) 779-7774

***Abstract*—A software tool which presents analytical solutions of Laplace's equation offers several advantages over numerical techniques, such as the opportunity to study parametric dependence of the solutions, and the ability to deal easily with derivatives and highly curved surfaces. The principal obstacles to the use of analytical solutions have been the difficulty of matching the shape to a solution, and the need to perform a large number of tedious symbolic calculations using algebra and calculus. These difficulties are overcome by a computer program which guides the user through a series of graphical menus to the correct problem statement. The solution is then furnished in a text editing window, and can be copied and exported to other programs, such as word processors and spreadsheets.**

INTRODUCTION

Laplace's equation forms the foundation of many fields of applied physics and engineering, and knowledge of its solutions is required of every undergraduate in these areas. It governs the steady state behavior of physical quantities such as temperature, voltage, magnetic fields, fluid flow, and pressure, and forms the starting point for the study of the dynamic behavior of the same quantities. It is also the introduction to potential theory, a field of applied mathematics with a long history and a reputation for long and difficult calculations. This combination of great importance and difficult mathematics makes Laplace's equation an ideal candidate for computer assisted engineering.

In many ways, the ideal approach to the solution of Laplace's equation is the analytical approach, which expresses the potential and its related quantities such as heat flux or electric field as a mathematical equation. The advantages of this approach are:

- *Parametric dependence is clear.* For example, the electric field around a point charge, q, has the form

$$E \sim \frac{q}{r^2}$$

which immediately displays the inverse relation for distance. Functional dependence is easily seen in the more complicated solutions also, giving the student insight into the behavior of the field.

- *Derivative variables are easily obtained.* Since the solution is an analytical function, it can be operated on with standard mathematical techniques to provide additional information. For example, the electric field can be obtained by differentiating the potential at any point in space, and the total electric charge on an object can be determined by integrating the electric field over a surface which encloses the object.
- *Many problems have already been solved.* Laplace's equation has been intensively studied for well over a century, because of its overwhelming importance to the physical sciences. There are literally thousands of analytical solutions for electrostatic problems which are available in the literature [1, 2, 3, 4], each for a different combination of shapes and boundary conditions.
- *Curved surfaces can be handled exactly.* The solution is carried out in terms of coordinate systems with curvilinear surfaces, so the solution is formulated exactly, not as rectangular or triangular approximations, like those of the finite element or finite difference techniques.

With all these advantages, however, practicing engineers rarely choose analytical solutions when they need to find electric fields. Instead, they choose numerical techniques such as finite differences or finite elements. Numerical methods require that all dimensions in the problem be locked into place before beginning the solution, so they can not be used to examine functional dependence without tedious (and carefully selected) repetition of the solution process. If derivative quantities such as heat flux and electric field are required, they must be obtained by numerical differentiation, which always introduces large errors into the solution, and therefore requires much finer detail in the solution for the potential. With all these problems, why are the numerical methods so popular?

The answer lies with the nature of the analytical solutions. They have a single drawback which is overwhelming to an engineer or scientist with an undergraduate education, and can be daunting even after years of graduate study. Analytical solutions of Laplace's equation require an

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immense amount of complicated mathematics. Even the simplest of them, which are usually taught at the Junior level, require facility with vector calculus. More useful solutions involve partial differential equations and differential geometry, which represents the highest level of mathematics reached by most engineers. The solution of a typical problem for a single sphere is likely to involve solving two or three ordinary differential equations with an infinite series of transcendental functions, patching them together with boundary conditions, solving a set of algebraic and transcendental equations for unknown coefficients, and integrating the solution over several dimensions. If these steps are completed without a mathematical error, the answer would be expressed as an infinite (and slowly converging) series of Legendre functions, each of which must be evaluated with an argument which is a trigonometric function. To get a numerical result, a program must be written to evaluate a large number (not known in advance) of the terms. With all of this additional work, ranging from computer programming to partial differential equations, it is not surprising that scientists and engineers who value their time turn to numerical packages using techniques such as finite elements, even though they lose much of the physical insight which comes with the analytical solution.

A more fundamental problem with the analytical approach lies in the way in which Laplace's equation is solved. Generally the problem arises in a context with a specific geometry and material properties, but the solutions are categorized by techniques. Thus one problem involving a sphere might be solved using separation of variables, while another would only respond to image charge techniques. An infrequent solver of Laplace's equation will not be expected to know the appropriate technique for each possible problem.

The present paper describes the development of an interactive computer program which will enable students (and also working scientists and engineers) to obtain analytical solutions to Laplace's equation for numerous shapes without the need to perform the complicated mathematics involved in the solution or to memorize a large number of solution techniques. The user interface will allow specification of the problem by means of graphical menus, and will provide the solution as an analytical expression in a text document.

THE SOLUTION TECHNIQUE

Designing a computer aided solution of Laplace's equation should begin by carefully tracing through the steps actually taken by a experienced engineer, and then replacing these steps as much as possible with the computer. As is often the case when a computer is being introduced to a problem, there are a number of steps which are taken for granted by an experienced practitioner, but which must be carefully described before the computer can proceed. As an example, consider the following problem, which is usually seen in the first course in electromagnetism.

A point charge, q, is placed a distance, d, above a conducting plane. Find the E field on the surface of the plane, and the force acting on the charge.

Unlike the approach taken in many textbooks, we assume here that the engineer really needs to know the value of the electric field and the force, and will not be satisfied with a mathematical expression. For example, the problem might involve the attachment of a toner particle to the paper in a laser printer, and a minimum force must be generated to preserve the integrity of the image. Predicting the operation of the laser printer will require that the numerical value of the electrostatic force be known for a given toner charge and location. In addition, reports on the work must be prepared, so the results must eventually be translated into charts and other documents.

The first step in solving this problem is usually drawing a sketch of the situation, like that shown in Fig. 1.

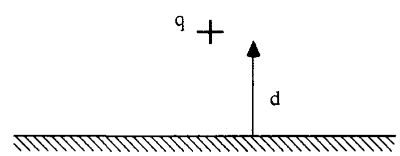


Figure 1. Sketch of geometry

An experienced solver of Laplace's equation recognizes this problem as one which can be solved using the method of images. To put it differently, the engineer has *memorized* a solution technique for this problem. The technique consists of placing a fictitious charge of equal and opposite magnitude below the surface of the conducting plane. The electric field at any point in the upper half space is then given by superposition of the fields from both charges as

$$\bar{E} = \frac{q_1}{4\pi\varepsilon_0 r_1^3}\bar{r}_1 + \frac{q_2}{4\pi\varepsilon_0 r_2^3}\bar{r}_2$$

This is a formal solution to the problem, but it is not directly useful to someone who needs to know the electric field, since it contains two distances which must first be calculated (r_1 and r_2), and two vectors which must be added. At this point, the engineer would find it necessary to write a program or spreadsheet to evaluate all the needed terms. Once the solution is obtained, the results have to copied into the report, along with the explanations and conclusions.

Thus the experienced solver of Laplace's equation actually recalls a memorized solution and customizes it to the problem at hand. The customization involves specifying the parameters, performing subsidiary calculations, plotting results, and preparing a report.

The goal of the software described here is to supply the 'memorized' solution, perform the subsidiary calculations, and enter the results into a word processing document. The availability of solutions allows a beginning student the same access to solutions as an expert. In addition, the built-in subsidiary calculations and formatted output spare the expert from much of the tedious and error-prone aspects of solving a complete problem.

In order to reach this goal, the solution procedure is divided into two parts, depending on the need for direct interaction with the user. All of the steps which do not require input from the user are done in advance, and collected into a catalog of problems which serves as a resource file. This catalog, which is prepared by an expert in field theory, contains both the problems and their solutions, including expressions for the potential and related quantities such as flux, and a description of the geometry and boundary conditions. This database functions as a catalog of pre-solved general solutions in which all of the difficult calculus and much of the algebra has already been done.

The second part of the solution procedure is carried out by means of a software tool for the student or engineer. This program, which incorporates the catalog described above, guides the user to the correct solution with a series of menus which present a choice of geometries and boundary conditions. The final selection leads to a graphical representation of the problem and a list of parameters which can be changed as desired to fit the desired problem. Once the parameters are selected, the program presents the solution as an analytical equation in a text document. The document can be edited, and the equation can be exported to other programs for further work.

The program adapts these general solutions to the particular case at hand, performs additional algebraic simplifications, and formats the equations for the desired output. It can reduce the equations to numerical values, if desired, and can also export them in a form useful for spreadsheets and word processors.

ADVANCE PREPARATION OF THE PROBLEM CATALOG

The heart of the project is the catalog of problems and solutions, which contains all the information needed to completely represent the behavior of a solution for a particular class of shapes and boundary conditions. This catalog must be prepared in advance by someone knowledgeable in electrostatics, and put into a form which can be accessed by the student or working engineer. In order to completely describe the problem, the catalog entry for each problem has been formulated as a data structure containing:

- Co-ordinate system description
- Number of distinct regions (topology)
- Material properties of each region
- Number of boundaries (including infinity, if appropriate)
- Shape of each boundary
- Locations of the boundaries
- Nature of constraint on each boundary (potential, gradient, combination)
- General form of the potential solution in each region.

All of this information is entered by an expert. Additional information can be calculated by symbolic mathematical programs. At the present time the only derived information is

- Electric fields in each region

Defining the problem

The general form of the information required is determined by the solution process for Laplace's equation. In order to make the requirements clearer, we will use a specific problem as an example of the information needed to completely specify and solve a problem. The problem involves two concentric spheres which are held at different potentials, as shown in Fig. 2.

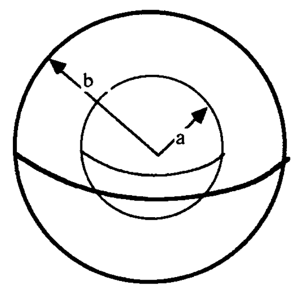


Figure 2. Two concentric spheres

One of the essential parts of the problem definition is the relation between the Cartesian coordinates and the natural coordinates of the problem, which has the form

$$x = x(u, v, w)$$
$$y = y(u, v, w)$$
$$z = z(u, v, w)$$

In problems with spherical symmetry, for example, we usually use the spherical coordinate system, which is related to the Cartesian system by the parametric equations:

$$x = r\cos\theta\cos\phi$$
$$y = r\cos\theta\sin\phi$$
$$z = r\sin\theta$$

These definitions are needed in the next step to find the gradient and other vector calculus results, and are also essential for a description of the boundaries and regions of the problem.

The second essential component of the problem definition is the specification of the boundary conditions which usually consists of several parts, such as the nature of the condition, its value at the boundary, and the location and shape of the boundary. In the example, the potential is specified to have the value V_1 volts at the inner sphere. This boundary is a sphere located at the origin, which is specified in the spherical coordinate system as

$$r=a$$

More complicated boundary conditions, involving several dimensions and combinations of potential and flux, can arise in advanced problems.

The third essential component of the solution for the potential is the form of the potential itself. In general, the solution has the form

$$\Phi = F_1 B_1 + F_2 B_2 + \ldots$$

where the B_i is the i^{th} boundary value (potential, charge density, etc.) and F_i is a coefficient which depends on the geometry and material properties. In our example, the general solution has the form

$$\Phi = \frac{\frac{b}{r} - 1}{\frac{b}{a} - 1} V_1 + \frac{\frac{b}{a} - \frac{b}{r}}{\frac{b}{a} - 1} V_2$$

so that the coefficients are given by

$$F_1 = \frac{\frac{b}{r} - 1}{\frac{b}{a} - 1} \qquad F_2 = \frac{\frac{b}{a} - \frac{b}{r}}{\frac{b}{a} - 1}$$

These coefficients are all that is needed to reconstruct the solution for given boundary values, so these are the expressions that are stored in the data structure.

Note that the symbols 'a' and 'b' represent the same parameters as in the definition of the boundary shape and location. The program includes appropriate means to make this connection.

These components of the data structure which makes up the problem statement are defined in a special program which allows for interactive entry and editing of the problem definition. This program, which is written in ANSI C for portability, is used by an expert in electrostatics to make up the catalog of problems. Once the problem definition is entered, it can be printed out, or exported to a second program which performs all of the calculus-based steps needed to complete the general solution.

Performing the calculus-based solution steps

The solution steps which require advanced mathematical manipulations are carried out in a second program written in Mathematica, a symbolic mathematics platform. This program, in three successive steps, defines the metric coefficients of the coordinate system, calculates the symbolic gradient, and then checks the result for consistency with Laplace's equation.

The first step is the calculation of the metric coefficients for the coordinate system, defined by

$$g_{uu}(u, v, w) = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2$$

$$g_{vv}(u, v, w) = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2$$

$$g_{ww}(u, v, w) = \left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2$$

For our spherical example, these become

$$g_{rr} = 1$$
$$g_{\theta\theta} = r^2$$
$$g_{\phi\phi} = r^2 \sin^2\theta$$

Once these have been calculated, the electric field can be obtained by evaluating

$$\mathbf{E} = -\nabla\Phi = -\frac{1}{\sqrt{g_{uu}}}\frac{\partial \Phi}{\partial u}\hat{i}_u - \frac{1}{\sqrt{g_{vv}}}\frac{\partial \Phi}{\partial v}\hat{i}_v - \frac{1}{\sqrt{g_{ww}}}\frac{\partial \Phi}{\partial w}\hat{i}_w$$

which in our example becomes

$$\mathbf{E} = -\frac{\partial \Phi}{\partial r}\hat{i}_r - \frac{1}{r}\frac{\partial \Phi}{\partial \theta}\hat{i}_\theta - \frac{1}{r \sin\theta}\frac{\partial \Phi}{\partial \phi}\hat{i}_\phi$$

One advantage of a symbolic mathematics language like Mathematica is the ability to perform automatic checks of the solution. This capability is used here to evaluate the divergence and curl of the electric field, to ensure that it satisfies the conditions

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \times \mathbf{E} = 0$$

which are required of any solution of Laplace's equation. If either of these expressions should be non-zero, there is a problem with the proposed solution of the problem. Usually, this indicates a mistake in the form of the solution in the problem statement.

If all the requirements for a solution are met, the Mathematica program writes out a symbolic solution file for the gradient associated with each coefficient of the potential solution.

Preparing the problem/solution package

The symbolic output file is then exported to a third program which merges the solution with the original problem statement file, and translates the merged data structure into a form suitable for accessing by the user program. At this point the data structure for the catalog entry of the problem is complete.

INTERACTION WITH THE USER

The problem catalog contains the results of all the difficult and time-consuming steps of the solution of Laplace's equation, but it must be made easily accessible before it can serve as a useful tool for a student or engineer. This access is provided by a user-oriented program called *Potentials...*™ which allows a user to select the problem based on its geometry, enter specific values of parameters, and then receive a text document containing the problem statement and solution as a symbolic analytical expression. The program is segmented internally into a kernel which provides all the mathematical and database capabilities, and a front end which handles the user interface. The kernel is written in ANSI C, and can be implemented on virtually any computer. At present, the front end has only a Macintosh version. The operation of this program is described below.

Selecting the problem

When *Potentials...* starts up, it first loads information on the physics of electrostatics, such as the permittivity of free space, the symbols which represent the potential and fields, and so forth. It then opens the problem selection window and waits for user input.

The first step is to select the primary element from the popup 'Geometry' menu (Fig. 3).

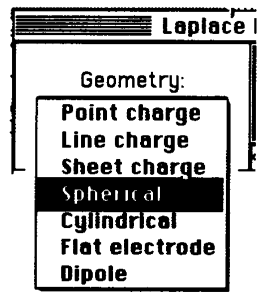


Figure 3. Selecting a geometry from the popup menu.

This menu contains virtually all the geometries seen in an introductory course in field theory , such as lines, circular cylinders, and dipoles. For the example of two concentric spheres, the appropriate choice ('Spherical') is shown highlighted in the figure.

Once the user selects a geometry class, the program displays a selection bar of icons representing the various problems in its database, as shown in Figure 4.

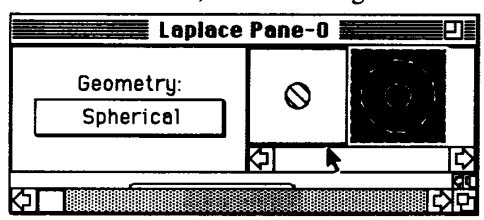


Figure 4. Selecting a problem from the scrolling icons.

In this example, concentric spheres are represented by the second icon, which is highlighted in the figure. Once it is selected, the program will load the problem statement and solution into memory, select default values for all of the parameters, and display a picture of the geometry along with a list of all the parameters which can be changed by the user. The complete problem window at this point is shown in Fig. 5.

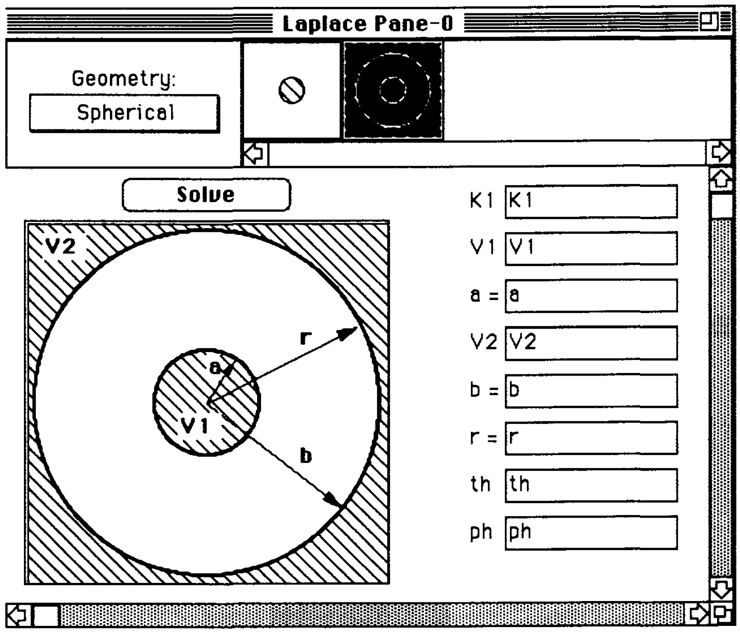


Fig. 5. The problem definition window

At this point the user can enter new values for any of the parameters, to bring the general solution into accord with the problem at hand. For example, an electrode can be grounded by setting its potential to zero, or a circular electrode can be enlarged by changing its diameter.

Once the parameters have the required values, the user's job is finished. The rest of the solution will be carried out by the program as soon as the 'Solve' button is pressed.

Performing the algebraic simplifications

When the 'Solve' button is pressed, the program first determines the current values for all of the parameters of the problem. It uses these values, along with the symbolic coefficients of the solutions for potential and electric field to form general expressions for the complete solution.

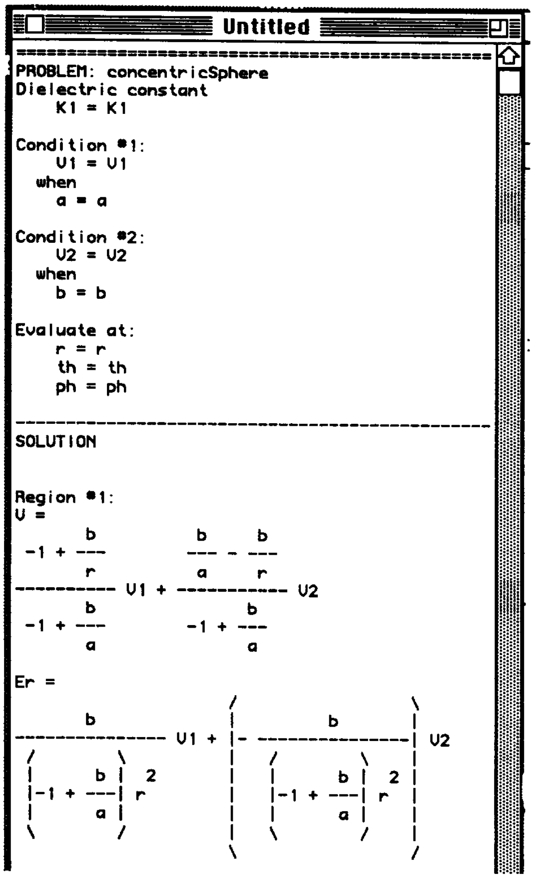
The program next attempts algebraic simplification of the symbolic expression. For example, any term with a factor of zero is removed, which eliminates coefficients associated with grounded electrodes. Other simplifications, such as a+0 -> a, are also performed. Finally, all expressions involving numbers (as opposed to literal symbols) are simplified by carrying out the indicated arithmetic operation. The result of this process is a set of simplified analytical expressions for the potential and electric fields. For the concentric sphere problem, these take the form

```
V = (((((-1 + (b)/(r)))))/((((-1 +
(b)/(a))))))*V1 + ((((((b)/(a) -
(b)/(r)))))/((((-1 + (b)/(a))))))*V2

Er = ((b)/(((((-1 + (b)/(a))*r^2)))))*V1
+ (-((b)/(((((-1 + (b)/(a))*r^2))))))*V2
```

Preparing the output

The unformatted symbolic expressions, even in their simplified form, are usually hard to interpret, so the program includes the ability to output the equations in a form similar to the standard representation of equations. If this output form is selected, the problem statement and its solution appear in a document window like the one shown in Fig. 6.



```
Untitled
PROBLEM: concentricSphere
Dielectric constant
    K1 = K1

Condition #1:
    U1 = U1
  when
    a = a

Condition #2:
    U2 = U2
  when
    b = b

Evaluate at:
    r = r
    th = th
    ph = ph

SOLUTION

Region #1:
U =
       b            b   b
 -1 + ---          --- - ---
       r            a   r
---------- U1 + ----------- U2
       b             b
 -1 + ---      -1 + ---
       a             a
Er =
                    /                 \
       b            |        b        |
--------------- U1 + |- ---------------| U2
 /          \        |  /          \   |
 |      b | 2        |  |      b | 2  |
 |-1 + ---| r        |  |-1 + ---| r  |
 |      a |          |  |      a |    |
 \        /          |  \        /    |
                     \                /
```

Fig. 6. The solution window

This window has standard text editing capabilities, so that notes, comments, or longer descriptions of the problem can be added by the user. The text can be saved in a file, and individual portions can be copied and pasted using the standard Macintosh techniques. For example, the formatted equation can be copied from the output window and pasted into a standard word processor..

The unformatted form of the output can also be obtained in a text editing window, and can be copied and pasted into other programs such as spreadsheets. As an example, the contour plot of potential around two unequal charges (Fig. 7) was prepared by copying the output of *Potentials...* into Mathematica™, and then calling for a contour plot.

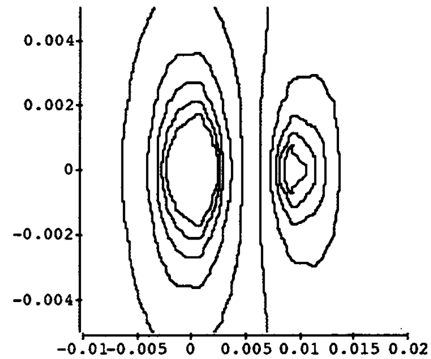


Fig. 7. Contour plot using output exported to Mathematica

CONCLUSION

The program described above is a first step toward the goal of providing computer-aided analytical solutions of Laplace's equation to students and working engineers. This goal is pursued by carrying out the difficult and time-consuming steps in advance under the supervision of an expert, and then transforming these results into a comprehensive catalog of solutions which can be accessed by a user with the standard undergraduate education in engineering. The access is structured to follow the practical needs of the engineer, so it follows a path based on the geometry of the situation rather than the method of solution. It is hoped that this approach will allow students and engineers to spend more time gaining physical insights into electrostatics, rather than spending time on mathematical details.

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