although the state trajectories, in general, were different. Each case was run 50 times and the ensemble averages were approximated by the numerical average of these 50 trials. In all cases the weighting coefficients for the control effort $\{B_i\}$ are unity, and the process noise variance is constant, but varies from case to case.

Numerical Results

The results of a seven stage stochastic process are described. Quantitative results are displayed for a terminal control problem, and the results of other simulations are discussed.

Terminal Control: In the terminal control problem, all state weighting coefficients are unity except the last one A_7 which is 100. This means, roughly, that the rms terminal error is ten times more important than the other quantities in the cost function.

Fig. 2 shows the results when the measurements are taken through a three level quantizer: the ensemble mean-square state and ensemble average of the (approximate) conditional covariance of the state are plotted as a function of time. For this case, the variance of the process noise is 0.2, and the quantizer switch points are at ± 1.

The most noticeable difference between the two control laws is that the one-measurement control acts to reduce the conditional covariance of the state estimate. Note that the ensemble average of the conditional covariance is about half the average of the conditional covariance for the open-loop control. The one-measurement control is able to reduce the conditional covariance by centering the conditional distribution of the measurement near the quantizer switch point, where a more accurate measurement can be obtained. This strategy is reflected in the curves for the mean-square value of the state which stays in the neighborhood of 1.0 (the switch point) for the one-measurement control but gradually goes to zero for the open-loop control. The control effort (not shown) for the one-measurement control is higher, and it requires a large control action at the last application to bring the state from the vicinity of the quantizer switch point to the origin.

The performance penalty of the open-loop-optimal feedback control over the one-measurement-optimal feedback control is 17.5 percent for this case. Other simulations revealed that the performance penalty ranged as high as 44 percent when observations were taken through a two level quantizer.

Other Simulations: Cost functions other than the terminal control type were simulated: the state deviations were weighted more heavily as time progressed, or else the weightings were constant. The performance advantage of the one-measurement control was always less than 10 percent in these cases. This arises from the fact that the one-measurement control tries to move the state around to gain information, but these movements are restricted by the relatively heavy weighting on the state deviations.

Thus a qualitative assessment, at least for linear systems and nonlinear measurements, is that incorporating future measurements in the control computations will yield the greatest return when the cost function is such that the state and/or control is free to reduce uncertainty in the estimate. In other situations, the open-loop control is quite attractive, especially because of its computational simplicity.

V. Conclusions

A new suboptimal stochastic control algorithm is presented which incorporates future measurments into the control computations. The two limiting forms of the algorithm are the well-known open-loop-optimal feedback control and the truly optimal stochastic control. This concept can be extended to simplify the computations for constrained controls, nonlinear plants, and nonquadratic cost criteria. A linear system with quantized measurements was simulated to compare the one-measurement-optimal feedback control to the open-loop-optimal feedback control. The greatest improvement over the open-loop algorithm occurs in those situations where the cost function gives the control and state some freedom to reduce the uncertainty in the state estimate.

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References

[1] S. Dreyfus, "Some types of optimal control of stochastic systems," *SIAM J. Control*, vol. 2, pp. 120–134, 1964.
[2] A. A. Fel'dbaum, "Dual control theory, pt. I," *Automation and Remote Control*, vol. 21, pp. 874–880, April 1961; pt. II, vol. 21, pp. 1033–1046, May 1961; pt. III, vol. 22, pp. 1–12, August 1961; pt. IV, vol. 22, pp. 109–121, September 1961.
[3] R. E. Curry, "Estimation and control with quantized measurements," Ph.D. dissertation, Dept. Aeron. and Astron., Massachusetts Institute of Technology, Cambridge, 1968; also M.I.T. Experimental Astronomy Lab. Tech. Rept. TE-23; revised and expanded in *Estimation and Control with Quantized Measurements*. Cambridge, Mass.: M.I.T. Press (to be published).

Control of an Amplifying Wave on an Infinite Continuum

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***Abstract*—Previous work on the space-sampled feedback control of continuum instabilities in flowing systems is extended. When the continuum over which control is desired is many wavelengths long, it is often convenient to construct the control system of many sampling stations, each approximately one wavelength in size. The stability of such a system is discussed, and a new type of instability is found which does not appear when the system is small in terms of a wavelength.**

Many systems described by wave equations, such as, boundary layers [1], confined fusion plasmas [2], and liquid jets [3], are subject to various types of instabilities under their normal operating conditions. Since these instabilities are often destructive, much effort has been expended in devising methods for improving the stability of these systems without sacrificing operating performance. A promising approach to this problem is space-sampled feedback control, in which the disturbances leading to instability are sampled over the entire spatial extent of the system, and spatially distributed forces are applied to attenuate these disturbances.

Continuum systems which satisfy wave equations may be described as spacelike or timelike [4]. In a spacelike system, waves may propagate in all directions, so that an instability once started, will eventually spread throughout the entire system. Such instabilities are often called absolute or nonconvective. The feedback stabilization of these instabilities in electromechanical systems and in plasmas has recently been reported in some detail [5]–[10].

In a timelike system, however, waves can propagate only in preferred directions. Physically, this situation usually arises when there is mechanical motion at speeds greater than the velocity of wave propagation. Common examples are supersonic flight and traveling wave amplifiers.

The instabilities which arise in timelike systems are called amplifying waves or convective instabilities. A previous paper [11] discussed the feedback control of an amplifying wave over a short distance, and reported an experimental attenuation of a disturbance which normally exhibits significant growth over a distance of a few wavelengths. In many cases of practical interest, however, significant growth of the instability occurs only on a system which is many wavelengths long. A successful control system requires that a

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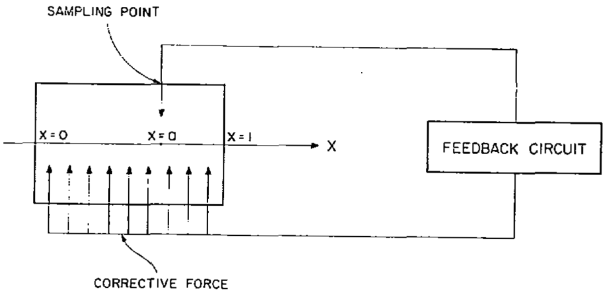

Fig. 1. Schematic representation of single sampling station in which disturbance measured at single point controls corrective force over entire length of section.

band of spatial frequencies be measurable and controllable. One way to satisfy this condition is to make each section shorter than a single wave length. Since this approach requires a large number of sampling stations, the present paper extends the results of [11] to cover space-sampled feedback on a system many wavelengths in extent.

I. System Equations

The system studied in [11], a liquid jet subject to a radial electric field, was chosen because it could be modeled quite well by a relatively simple wave equation which is representative of most timelike systems, and because it could be easily produced in the laboratory.

In the absence of feedback, the liquid jet can be described by the nondimensional equation [11]

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\delta = \alpha^2 \frac{\partial^2 \delta}{\partial x^2} + N\delta. \tag{1}$$

The term on the left accounts for the inertia of the jet while the terms on the right represent the restoring force of surface tension (α^2) and a destabilizing force proportional to displacement, due to the electric pressure (N) on the surface. The quantity δ here represents the amplitude of the disturbance on the jet, and x and t are real space and time variables.

If the velocity of the jet is greater than the wave propagation velocity ($\alpha < 1$), this system is timelike and exhibits amplifying waves which have been discussed elsewhere in detail [3], [12]. In the work reported in [11], these amplifying waves were controlled by measuring the disturbances at discrete points and applying a uniform correcting force to the jet in the neighborhood of each of these points. The feedback system can, therefore, be treated as a series of independent sampling sections which are individually represented by Fig. 1.

If the sampled feedback force is included, the nondimensional equation which describes the system within each section is [11]

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)^2 \delta(x,t) = \alpha^2 \frac{\partial^2 \delta(x,t)}{\partial x^2} + N\delta(x,t) - M\delta(x = a,t) \tag{2}$$

where M is the feedback gain, and a is the sampling point.

Since this is a one-dimensional timelike system, all disturbances propagate in only one direction (the direction in which the jet moves downstream). The boundary conditions which influence the disturbance must therefore be applied at the upstream end of each sampling section, because disturbances applied at the downstream side would propagate farther downstream and never convey information into the section. These conditions are analogous to initial conditions which affect the behavior of the system only for times after the time of application. The disturbance generated by these entrance conditions is then carried through the section to the downstream end where it serves as the upstream boundary condition for the succeeding section. The system is convectively unstable if the magnitude of the disturbance grows without bound as it is swept downstream from section to section; however, the magnitude at any fixed point remains bounded.

The sinusoidal steady state disturbance at the outlet of the nth section is given by successive solutions of the wave equation [11] in terms of the initial disturbance

$$\delta_0 = \delta(x = 0,t) \tag{3}$$

and its spatial derivative

$$\delta_0' = \frac{\partial \delta_0(x = 0,t)}{\delta x} \tag{4}$$

as

$$\begin{bmatrix} \delta_n \\ \delta_n' \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}^n \begin{bmatrix} \delta_0 \\ \delta_0' \end{bmatrix} \tag{5}$$

where

$$KA_{11} = \gamma c(1)e(1) - \beta c(a)e(a) + ik_r[\gamma s(1)e(1) - \beta s(a)e(a)]$$

$$KA_{12} = \gamma s(1)e(1) - \beta[s(a)e(a) + s(1 - a)e(1 + a)]$$

$$KA_{21} = \gamma(k_r^2 + k_i^2)s(1)e(1)$$

$$KA_{22} = \gamma c(1)e(1) - \beta c(1 - a)e(1 + a) - ik_r[\gamma s(1)e(1) - \beta s(1 - a)e(1 + a)]$$

and

$$K = 1 - \beta\{[c(a) + ik_r s(a)]e(a) - 1\}$$

$$\beta = M/(N + \omega^2)$$

$$\gamma = 1 + \beta$$

$$e(x) = \exp(-ik_r x)$$

$$c(x) = \cosh k_i x$$

$$s(x) = (\sinh k_i x)/k_i$$

$$k_r = \omega/(1 - \alpha^2)$$

$$k_i = \sqrt{N(1 - \alpha^2) - (\alpha\omega)^2}/(1 - \alpha^2).$$

II. The Stability Criterion

The nature of the response far from the point of excitation can be determined by considering the limit of (5) as n becomes very large. If the response is bounded, this limit must be bounded. The nature of the limiting response can be determined most easily by transforming the transfer matrix to its canonical form

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where λ satisfies the equation

$$\begin{bmatrix} A_{11}-\lambda & A_{12} \\ A_{21} & A_{22}-\lambda \end{bmatrix} = 0. \tag{6}$$

If the magnitude of either eigenvalue is greater than unity, the limiting response, which is proportional to the eigenvalues raised to a very large power, will be unbounded. The condition for spatial stability of the jet is then

$$|\lambda_1| < 1$$
$$|\lambda_2| < 1. \tag{7}$$

A special case is the uncontrolled jet ($M = 0$) excited by a constant disturbance. The response matrix under these conditions is

$$\begin{bmatrix} \cosh k_i & \sinh k_i/k_i \\ k_i \sinh k_i & \cosh k_i \end{bmatrix}.$$

In canonical form, this matrix may be written

$$\begin{bmatrix} e^{k_i} & 0 \\ 0 & e^{-k_i} \end{bmatrix}.$$

This represents the growing and decaying waves normally found on the jet. For a real value of k_i, one of the eigenvalues will be greater than unity, indicating spatial instability. Thus this new criterion is identical to the usual stability criterion for timelike systems based on the dispersion relation of the free waves, and given by [8], [9]

$$k_i = 0. \tag{8}$$

III. Spatial Instabilities with Sampled Feedback

To delimit the stable operating range, we choose appropriate values of the parameters a, α, N, and M, and evaluate the eigenvalues of the response matrix for all frequencies. If neither eigenvalue attains a magnitude greater than unity, the chosen operating point lies in the stable region. By repeating this procedure at different operating points, a stable operating region in the M–N plane can be mapped out for each value of a and α.

Amplifying Wave

The preceding search procedure has revealed three types of spatial instabilities on the feedback controlled jet. The first is the growing wave normally present on the jet. The growth rate of this disturbance is largest at low frequencies, and vanishes as the frequency is raised above a definite cutoff frequency given by

$$\omega_{co} = \frac{\sqrt{N(1 - \alpha^2)}}{\alpha}. \tag{9}$$

To control this instability, the feedback system must furnish a restoring force over the frequency range

$$0 < \omega < \omega_{co} \tag{10}$$

with sufficient amplitude to counteract the growth. A plot of the feedback gain needed to overcome the growing wave for different growth rates (Fig. 2) shows that the condition for stability is very nearly

$$M > N \tag{11}$$

in agreement with the ideal continuum feedback case discussed in [11].

Spatial Overstability

As the feedback gain at low frequencies is increased above the value used to control the growing wave, the jet remains stable over a certain range of gain, and then becomes unstable in a new mode, unrelated to the original one. This new mode is a spatial overstability in which the feedback system overcompensates for the disturbance detected at the sampling point. Viewed in the laboratory frame, this disturbance occurs at zero frequency and has a wavelength of approximately two section lengths. The threshold gain for this instability is shown in Fig. 3.

The position of the sampling point has a strong effect on the spatial overstability, with the threshold gain decreasing as the sampling point is moved toward the entrance of the station. This is to be expected, since a sampling point near the entrance causes the feedback force to be exerted downstream for a longer fraction of the time the jet spends in the section, thus increasing the amount of overcontrol.

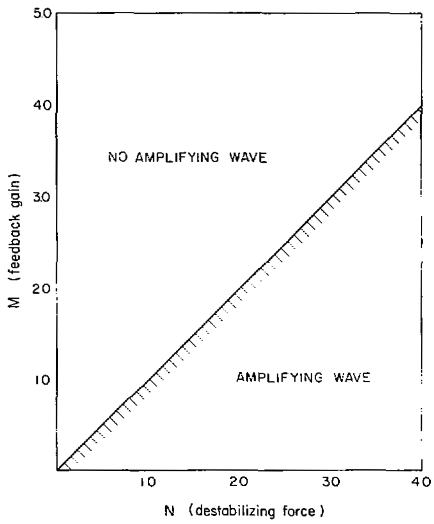

Fig. 2. Minimum feedback gain is needed to control amplifying wave, ($\alpha = 0.1$, $a = 0.5$).

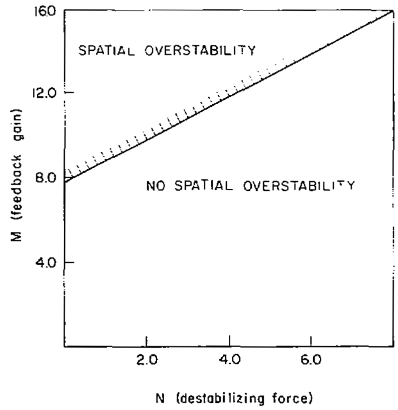

Fig. 3. Excessive feedback gain causes spatial overstability ($\alpha = 0.1$, $a = 0.5$).

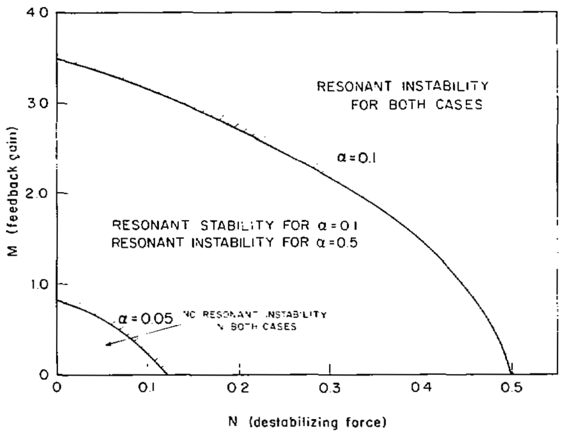

Fig. 4. Excessive feedback gain may also cause resonant spatial instability at frequencies near pole of response. Increasing tension α will enlarge stable region ($a = 0.5$).

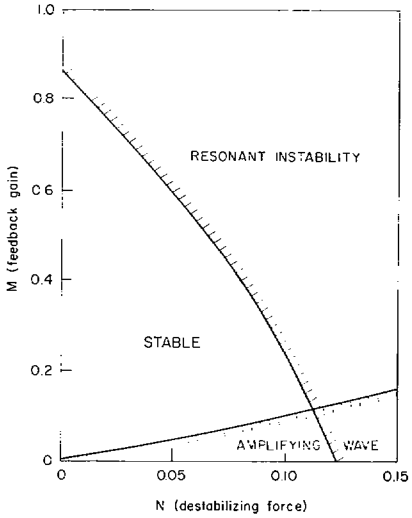

Fig. 5. Stable operation region when tension force is weak (α = 0.1).

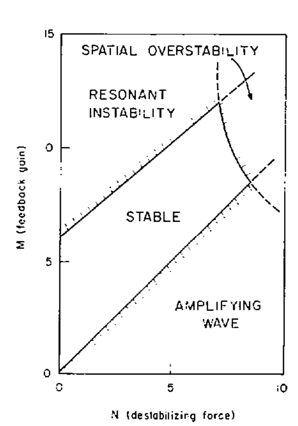

Fig. 6. Stable operation region when tension force is large (α = 0.5).

Resonant Instability

A third possible mode of spatial instability occurs because the feedback loop exhibits a time lag on the order of the transit time of the jet. A spatial instability of this type is called a resonant instability since it occurs when operating near a pole of the response [11]. The resonant instability, which occurs at a relatively high frequency, is opposed by the tension of the jet, and the threshold gain is therefore a strong function of α. The position of the sampling point also affects this instability, since moving the pickup toward the exit of the section increases the phase shift and lowers the threshold gain for instability. A plot of the threshold gain for two values of α with midpoint sampling ($a = 1/2$) shows the threshold gain decreasing as N increases and also as α decreases (Fig. 4).

The Stable Region of Operation

So far, the various spatial instabilities have been discussed individually, and the separate criteria for stability of the usual growing wave, the spatial overstability, and the resonant instability have been formulated. To ensure complete control of the instability, the operating point must lie in the stability regions of all three disturbances. This stable region can be determined by superposing the stable regions of all three modes.

The combined stability criteria for the case $a = 0.5$ (midpoint sampling) $\alpha = 0.1$ (Fig. 5) indicates that the stability for this case is determined by the original growing wave and the resonant instability, since the spatial overstability exhibits a higher threshold gain than the resonant instability for all values of N.

If the tension term α, is large, however, the threshold gain for resonant instability is increased, while the other two unstable modes are little affected. The combined stability diagram for $\alpha = 0.5$ (Fig. 6) shows that the spatial overstability now limits the allowed feedback gain over most of the range.

IV. Discussion

Three types of spatial instability are possible on a convective wave system with an infinite number of space-sampled feedback stations. The usual growing wave and the resonant instability are similar to instabilities which appear when the feedback system has a few stations. The spatial overstability, however, appears only when the number of stations is large and may be the limiting factor for some operating regimes. Despite these instabilities, a stable region of operation was shown to exist for all of the cases studied.

The variety of ways in which instability may occur in this system is due not only to the inherent instability in the uncontrolled system, but also to the nature of the feedback system employed. All of the results presented here are for the special case in which the sampling point is located at the center of each section, and the measured variable controls the force upstream (feedback) and downstream (feed forward) of the sampling point. The search procedure showed that if measurements were made much farther downstream, thus emphasizing the feedback portion of the control system, then there was no stable region of operation against the resonant instability. Emphasizing the feed forward aspects, on the other hand, lends to spatial overstability for all values of gain. Thus, a combination of feedback and feed forward control appears to be needed to ensure completely stable operation.

These results show the possibility of using a space-sampled feedback system to control a slowly growing amplifying wave on a very large system. The feedback system was chosen for simplicity and does not represent an optimum design for this particular instability. One obvious improvement would be the incorporation of a low pass filter in the feedback loop which would serve to suppress feedback at the resonant frequency of the transfer function. This would help stabilize the jet against the resonant instability. Another possible refinement is space tapering of the force application and sampling. This would decrease the sensitivity of the system to both the spatial overstability and the resonant instability by integrating over the entire length of the section to detect the disturbance and by decreasing the force applied far from the sampling point.

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References

[1] H. Schlichting, *Boundary Layer Theory.* New York: Pergamon, 1955, ch. 16.
[2] A. Jeffry and T. Taniuti, *MHD Stability and Thermonuclear Containment.* New York: Academic Press, 1966.
[3] J. R. Melcher, *Field Coupled Surface Waves.* Cambridge, Mass.: M.I.T Press, 1963.
[4] R. Courant and D. Hilbert, *Methoden der Mathematischen Physik,* vol. 2. Berlin: Springer, 1937.
[5] J. R. Melcher, "Control of a continuum electromechanical instability," *Proc. IEEE,* vol. 53, pp. 460–473, May 1965.
[6] ——, "An experiment to stabilize an electromechanical continuum," *IEEE Trans. Automatic Control,* vol. AC-10, pp. 466–469, October 1965.
[7] ——, "Continuum feedback control of a Rayleigh–Taylor type instability," *Phys. Fluids,* vol. 9, pp. 2085–2094, November 1966.
[8] ——, "Continuum feedback control of instabilities on an infinite fluid interface," *Phys. Fluids,* vol. 9, pp. 1973–1982, October 1966.
[9] V. V. Arsenin *et al.*, "Suppression of cyclotron instability of a rarefied plasma with the aid of a feedback system," *JETP Letters,* vol. 8, pp. 69–72, July 1968.
[10] V. V. Arsenin and V. A. Chuyanov, "A possible stabilization of the trough-shaped plasma instability through a feedback system," *Soviet Phys. "Doklady,"* vol. 13, pp. 570–571, 1968.
[11] J. M. Crowley, "Stabilization of a spatially growing wave by feedback," *Phys. Fluids,* vol. 10, pp. 1170–1177, June 1967.
[12] ——, "Growth and excitation of electrohydrodynamic surface waves," *Phys. Fluids,* vol. 8, pp. 1668–1676, 1965.